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von Kármán–Howarth relationship for helical magnetohydrodynamic flows

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We derive an exact equation for homogeneous isotropic magnetohydrodynamic (MHD) turbulent flows with nonzero helicity; this result is of the same nature as the classical von Kármán–Howarth (VKH-HM) formulation for the kinetic energy of turbulent fluids. Helical MHD is relevant to the astrophysical flows such as in the solar corona, or the interstellar medium, and in the dynamo problem. The derivation involves the new writing of the general form of tensors for that case, for either vectors or (pseudo)axial vectors. It is shown that, for general third-order tensors, four generating functions are needed when taking into account the nonmirror invariance of helical fluids, instead of two as in the fully isotropic case. The new equation obtained, denoted by VKH-HM, links the dissipation of magnetic helicity to the third-order correlations involving combinations of the components of the velocity, the magnetic field, and the magnetic potential. Finally, in the long-time and nonresistive limit, this relationship leads to a linear scaling with separation of the third-order tensor, correlating the two normal components of the electromotive force and of the magnetic potential.

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I. INTRODUCTION

In turbulent flows, the existence of rugged invariants I_0 that survive truncation of the nondissipative equations is thought to play an important dynamical role, and may lead to nontrivial behavior when more than one such invariant is present, as already noted by Onsager [1]. Indeed, an inverse cascade of kinetic energy to large scales in two space dimensions can be predicted for Navier-Stokes turbulence on the basis of an argument stemming from statistical mechanics; it uses the fact that both energy and enstrophy are conserved in the inviscid case (see, e.g., Kraichnan and Montgomery [2] for a review). Such invariants cascade with a Fourier spectrum $I_0(k)$, where k is the wave number, assuming isotropy; the cascade is either to small scales or to large scales and takes place at a rate ϵ_{I_0} prescribed by the injection mechanism into the system. Dimensional analysis in the manner of Kolmogorov then leads to a prediction for power-law spectra $I_0(k)$, e.g., the $k^{-5/3}$ kinetic energy spectrum for Navier-Stokes turbulence. Such Kolmogorov spectra are of a phenomenological nature except in the case of weak turbulence involving fast waves (see, e.g., Refs. [3,4]).

On the other hand, the presence of the invariants I_0 allows for the derivation of *exact* equations. For example, von Kármán and Howarth [5] derived the relationship between the rate of energy dissipation ϵ_{v^2} and the third-order correlation functions of the velocity field $\mathbf{v}$. Similar exact equations have been obtained for the variances of physical fields at several instances, e.g., for the variance of a passive scalar [6], for that of the magnetic potential in two-dimensional incompressible magnetohydrodynamic (MHD) [7], for the total (kinetic plus magnetic) energy $E_T=\langle \mathbf{v}^2+\mathbf{b}^2\rangle/2$ and total velocity-magnetic field correlation $H_C=\langle \mathbf{v}\cdot\mathbf{b}\rangle/2$ [8,9] in two- and three-dimensional incompressible MHD (where $\mathbf{v}$ is the velocity, $\mathbf{b}=\nabla\times\mathbf{a}$ is the magnetic induction, and $\mathbf{a}$ the magnetic potential), and for the kinetic helicity $H_V=\langle \mathbf{v}\cdot\boldsymbol{\omega}\rangle/2$ (where $\boldsymbol{\omega}=\nabla\times\mathbf{v}$ is the vorticity) for three-dimensional incompressible Navier-Stokes flows [10] (see also Refs. [11]).

Another invariant of the MHD equations for zero magnetic diffusivity [12] is the magnetic helicity

$$H_M=\frac{1}{2}\langle \mathbf{a}\cdot\mathbf{b}\rangle.$$

It is known to play an important dynamical role, e.g., in the solar corona—in flares and in coronal mass ejections [13,14]—and in the dynamo problem, i.e., the generation of magnetic fields through turbulent motions. Magnetic fields are observed in the core of the earth and planets, in the sun and stars, and in the interstellar and intergalactic media. Dynamos have been extensively studied theoretically and numerically, starting with the pioneering works of Parker [15] and Steenbeck *et al.* [16] in the former case and of Bullard and Gellman [17] in the latter. More recently, the dynamo effect in turbulent flows is being studied in the laboratory as well [18,19] in the specific configuration corresponding to two counter-rotating disks. The role of magnetic helicity in the dynamo mechanism arises in the nonlinear saturation of the so-called α effect, i.e., in the saturation of the growth of a magnetic field at a given (large) scale because of the presence of helicity (kinetic at first, kinetic and magnetic at later stages of the instability) at small scale. It was conjectured in Ref. [20] and shown—using second-order closures in Ref. [21], as well as direct numerical simulations in Refs.

[22,23]—that H_M undergoes an inverse cascade to large scales. Later works confirmed these findings, and this cascade is also known to persist for supersonically driven flows [24], showing that even in the interstellar medium where compressibility plays an important role because of, e.g., supernovae blast waves sweeping sporadically the gas, an α-like dynamo is plausible. This saturation mechanism is nonlinear and does not involve the later saturation of the growth of the *total* magnetic energy, a saturation which can be linked directly to the magnetic Reynolds number (a recent study of this point can be found in Brandenburg and Sarson [25]). Moreover, the heating of the solar corona, through dissipative processes, is linked to the reversal of the magnetic dynamo field, a process in which the conservation of magnetic helicity plays an essential role [13,26]. Note also that there is a resurgence of interest in helical flows in neutral fluids (see, e.g., Ref. [27] and references therein).

We thus propose to derive in this paper the exact equation that takes into account the fact that the induction equation conserves H_M (a simplified version in the one-dimensional case is done by Galtier *et al.* [28]). In so doing, we shall need to write from first principles the complete form of the second-order (see also Ref. [29]) and third-order tensors when the mirror symmetry is broken, when tensors involve different fields (namely, here the magnetic potential, the magnetic induction, and the velocity), and taking into account the fact that the magnetic induction is an axial vector. This is performed following the approach of Batchelor [30], and is shown in the Appendix for clear readership of the remainder of the paper. We give basic definitions in Sec. II, and we derive, in Sec. III, the dynamical von Kármán–Howarth equation for magnetic helicity, which we call VKH-HM. Finally, Sec. IV is the conclusion.

II. KINEMATICS OF HELICAL FLOWS

A. Definitions

We consider fluctuating fields in incompressible MHD flows; the zero-mean velocity $\mathbf{v}(\mathbf{x})$ and magnetic field $\mathbf{b}(\mathbf{x})$ are, respectively, the proper and pseudo(axial) vectors, with $\mathbf{x}$ the Cartesian position vector relative to some fixed origin. In the context of three-dimensional MHD turbulence, the magnetic helicity H_M is a pseudoscalar, interpretable as a measure of knottedness of the magnetic field lines [31,32].

The (unsymmetrized) magnetic helicity correlation tensor

$$\tilde{R}_{ij}^{H_M}(\mathbf{x},\mathbf{x}')=\langle a_i(\mathbf{x})b_j(\mathbf{x}')\rangle$$

is a pseudotensor, whereas the magnetic potential $\mathbf{a}(\mathbf{x})$ is a true vector; $\mathbf{x}'$ denotes the displaced position $\mathbf{x}+\mathbf{r}$ and brackets denote the ensemble average or equivalently, under an ergodic hypothesis, large-space or long-time average [30]. Finally, because of homogeneity, $\tilde{R}_{ij}^{H_M}(\mathbf{x},\mathbf{x}')=\tilde{R}_{ij}^{H_M}(\mathbf{r})$.

B. Kinematical construction of tensors

In the case of homogeneous turbulence which has spherical symmetry but does not have reflexional symmetry, i.e., for isotropic (or skew-isotropic) turbulence, as in the presence of kinetic or magnetic helicity, the structure of the two-point correlation tensors is more complex than in the full isotropic case, i.e., isotropic and reflexion (or mirror) symmetric cases. Their general form includes terms based on additional fundamental invariants, such as

$$\epsilon_{ij\ell}c_i d_j r_\ell\,,$$

in the expression of the mean value products of vector (or pseudovector) components in the directions of $\mathbf{c}$ and $\mathbf{d}$, the directed distance $\mathbf{r}$ describing the configuration formed by the two points at which the correlations are studied. These invariants preserve the rotational symmetry but change sign when the three-vector configuration $(\mathbf{c},\mathbf{d},\mathbf{r})$ is reflected at any point or plane [33]. In the above relationship, as usual, $\epsilon_{ij\ell}$ denotes the unit alternating tensor. Thus, for isotropic turbulence, one can note that the second-order two-point correlation tensor is no longer index symmetric.

The second-order correlation tensors have been dealt with in Ref. [29], and the main results of these authors are recalled in the first section of the Appendix. A thorough derivation of the general form of the homogeneous third-order tensors for incompressible isotropic but helical fields is obtained in the second section of the Appendix. For completeness, we restate in this section some of the tensorial objects needed to obtain the VKH-HM equation.

The second-order correlation tensor for the magnetic potential is a *proper* (true) tensor which, assuming homogeneity and isotropy, reads

$$R_{ij}^{aa}(\mathbf{r})=\langle a_i(\mathbf{x})a_j(\mathbf{x}')\rangle=A^{aa}(r)r_ir_j+B^{aa}(r)\delta_{ij}+\tilde{C}^{aa}(r)\epsilon_{ij\ell}r_\ell\,, \tag{1}$$

where $\tilde{C}^{aa}(r)$ is a pseudoscalar function and all the tensor coefficients are even functions of the distance $r=|\mathbf{r}|$ due to spherical symmetry.

Magnetic helicity is related to the term proportional to $\tilde{C}^{aa}(r)$. Indeed, the antisymmetric part of the tensor $\tilde{C}^{aa}(r)\epsilon_{ij\ell}r_\ell$ can also be written as [29,34]:

$$\epsilon_{ij\ell}\frac{\partial}{\partial r_\ell}\phi(\mathbf{r}),$$

where $\phi(\mathbf{r})$ is an even pseudoscalar function of $\mathbf{r}$, such that $\phi(\mathbf{r}=0)=H_M$ [see Eq. (A8)].

Using the Coulomb gauge, two generating functions remain linked to the magnetic helicity of the flow:

$$\frac{R_{ij}^{aa}(\mathbf{r})}{\overline{a^2}}=f^{aa}(r)\delta_{ij}+\frac{r}{2}\frac{\partial f^{aa}(r)}{\partial r}P_{ij}(r)+\epsilon_{ij\ell}\frac{r_\ell}{r}\tilde{s}^{aa}(r); \tag{2}$$

as usual

$$P_{ij}(r)=\delta_{ij}-(r_ir_j)/r^2$$

is the incompressibility projector $\overline{a^2}f^{aa}(r)$ is the (dimensionalized) longitudinal correlation function of the magnetic potential, and $\overline{a}$ is the rms magnetic potential. It has been necessary in the above equation to introduce as well a new function $\tilde{s}^{aa}(r)$, with

$$R_{23}^{aa}(r)=r\tilde{C}^{aa}(r)\epsilon_{23L}\equiv\overline{a^2}\epsilon_{23L}\tilde{s}^{aa}(r),$$

where the subscript L refers to the longitudinal (along $\mathbf{r}$) direction, while the subscripts 2 and 3 refer to the two remaining transverse directions in the three-dimensional space.

An analogous definition holds for the magnetic helicity pseudotensor with

$$\tilde{R}_{ij}^{H_M}(\mathbf{r})=\langle a_i(\mathbf{x})b_j(\mathbf{x}')\rangle=\epsilon_{j\ell m}\frac{\partial R_{im}^{aa}(\mathbf{r})}{\partial r_\ell}, \tag{3}$$

the derivative being carried on the magnetic field. It can also be written as

$$\frac{\tilde{R}_{ij}^{H_M}(\mathbf{r})}{c_2}=\tilde{f}^{ab}(r)\delta_{ij}+\frac{r}{2}\frac{\partial\tilde{f}^{ab}(r)}{\partial r}P_{ij}(r)+\epsilon_{ij\ell}\frac{r_\ell}{r}s^{ab}(r), \tag{4}$$

where $c_2=\overline{ab}$ is a (scalar) constant, with $\overline{b}$ the (scalar) rms magnetic amplitude. Note that we generally indicate the pseudocharacter of a function or a tensor with a tilde symbol except for H_M, the magnetic helicity itself.

The third-order correlation tensor needed to derive the VKH-HM equation is the following pseudotensor involving the velocity, the magnetic field, and the magnetic potential:

$$\begin{aligned}\Phi_{ij\ell}^{vba}(\mathbf{r})=\langle v_i(\mathbf{x})b_j(\mathbf{x})a_\ell(\mathbf{x}')\rangle&=\tilde{a}_{11}^{vba}(r)r_\ell\delta_{ij}+\tilde{a}_{12}^{vba}(r)r_j\delta_{i\ell}\\&+\tilde{a}_{13}^{vba}(r)r_i\delta_{j\ell}+\tilde{a}_{3}^{vba}(r)r_ir_jr_\ell+b_{23}^{vba}(r)r_i\epsilon_{j\ell m}r_m\\&+b_{22}^{vba}(r)r_j\epsilon_{\ell im}r_m+b_{21}^{vba}(r)r_\ell\epsilon_{ijm}r_m,\end{aligned} \tag{5}$$

where the seven coefficients appearing in this general definition are even functions of the distance r, as for the second-order correlation tensor. The incompressibility constraint leads to relationships between these coefficients and turns to a formulation of the tensor involving only four generating functions (see details in the Appendix).

III. A von KÁRMÁN EQUATION FOR MAGNETIC HELICITY

A. The derivation

We now move on to the dynamics of magnetic helicity in homogeneous MHD turbulent flows. The incompressible MHD equations for the magnetic potential $\mathbf{a}$ write

$$\partial_t\mathbf{a}=\mathbf{v}\times\mathbf{b}+\eta\Delta\mathbf{a}, \tag{6}$$

using the Coulomb gauge $\nabla\cdot\mathbf{a}=0$; in the above equation, η is the magnetic diffusivity. In MHD, this equation is coupled to the velocity equation, the latter not being needed to prove that magnetic helicity is an invariant of the flow; henceforth, we will not need the velocity equation to derive the exact VKH-HM law for magnetic helicity.

The equation for the time evolution of the second-order correlation pseudotensor $\tilde{R}_{ij}^{H_M}(\mathbf{r})=\langle a_i(\mathbf{x})b_j(\mathbf{x}')\rangle$ results from Eq. (6), written at point $\mathbf{x}$ and the induction equation written at point $\mathbf{x}'=\mathbf{x}+\mathbf{r}$. This leads to

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{R}_{ij}^{H_M}(\mathbf{r})&=\epsilon_{i\ell m}\Phi_{\ell mj}^{vbb}(\mathbf{r})+\partial_{r_k}[\Phi_{jki}^{vba}(-\mathbf{r})-\Phi_{kji}^{vba}(-\mathbf{r})]\\&+2\eta\frac{\partial^2}{\partial r_k^2}\tilde{R}_{ij}^{H_M}(\mathbf{r}),\end{aligned} \tag{7}$$

where the form of the $\Phi_{ij\ell}^{\cdot\cdot\cdot}$ tensors are derived in the Appendix for the general case [see Eq. (A9)], and $\partial_{r_k}=\partial/\partial r_k$.

Note that both (vba) and (vbb) tensors appear in the above equation. However, the proper tensor $\Phi_{\ell mj}^{vbb}(\mathbf{r})$ can be derived from the pseudotensor $\Phi_{\ell mj}^{vba}(\mathbf{r})$ as follows:

$$\begin{aligned}\epsilon_{i\ell m}\Phi_{\ell mj}^{vbb}(\mathbf{r})&=\epsilon_{i\ell m}\epsilon_{jpq}\partial_{r_p}\Phi_{\ell mq}^{vba}(\mathbf{r})\\&=r^2\delta_{ij}\left[\frac{1}{r}\frac{\partial(\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba})}{\partial r}+2\frac{\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba}}{r^2}\right]\\&-\frac{r_ir_j}{r}\frac{\partial(\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba})}{\partial r}+\epsilon_{ijm}r_m\\&\times\left[r\frac{\partial(b_{23}^{vba}+b_{22}^{vba})}{\partial r}+3(b_{23}^{vba}+b_{22}^{vba})\right.\\&\left.-2b_{21}^{vba}\right].\end{aligned} \tag{8}$$

The tensor coefficients $\tilde{a}_{\cdot\cdot}^{\cdot\cdot\cdot}(r)$ are pseudoscalar functions, while the $b_{\cdot\cdot}^{\cdot\cdot\cdot}(r)$ coefficients are scalar. Expressing all the third-order tensors as derived in the Appendix, the nonlinear terms in Eq. (7) write, after lengthy but straightforward algebraic computations:

$$\begin{aligned}&\epsilon_{i\ell m}\Phi_{\ell mj}^{vbb}(\mathbf{r})+\partial_{r_k}[\Phi_{jki}^{vba}(-\mathbf{r})-\Phi_{kji}^{vba}(-\mathbf{r})]\\&=2\delta_{ij}[r\partial_r(\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba})+2(\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba})]\\&-2\frac{r_ir_j}{r}\partial_r(\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba})+2\epsilon_{ijm}r_m[r\partial_r(b_{22}^{vba}+b_{23}^{vba})\\&+3(b_{22}^{vba}+b_{23}^{vba})-2b_{21}^{vba}],\end{aligned}$$

which, using the incompressibility condition [see Eq. (A22) in the Appendix with $d=3$]

$$b_{23}^{vba}+b_{22}^{vba}=4b_{21}^{vba}+r\partial_rb_{21}^{vba},$$

simplifies to

$$\begin{aligned}&\epsilon_{i\ell m}\Phi_{\ell mj}^{vbb}(\mathbf{r})+\partial_{r_k}[\Phi_{jki}^{vba}(-\mathbf{r})-\Phi_{kji}^{vba}(-\mathbf{r})]\\&=2\delta_{ij}\frac{1}{r}\partial_r[r^2(\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba})]-2r_ir_j\frac{1}{r}\partial_r(\tilde{a}_{13}^{vba}-\tilde{a}_{12}^{vba})\\&+2\epsilon_{ijm}r_m\frac{1}{r}\partial_r\left[\frac{1}{r^2}\partial_r(r^5b_{21})\right].\end{aligned}$$

On the other hand, the linear term is written as

$$\frac{\partial^2}{\partial r_k}2\tilde{R}_{ij}^{H_M}(\mathbf{r})=r_i r_j\left[r\partial_r\left(\frac{1}{r}\partial_r\tilde{A}^{ab}\right)+\frac{7}{r}\partial_r\tilde{A}^{ab}\right]+\delta_{ij}\left[2\tilde{A}^{ab}+r\partial_r\left(\frac{1}{r}\partial_r\tilde{B}^{ab}\right)+\frac{3}{r}\partial_r\tilde{B}^{ab}\right]+\epsilon_{ijm}r_m\left[r\partial_r\left(\frac{1}{r}\partial_r C^{ab}\right)+\frac{5}{r}\partial_r C^{ab}\right].$$

Finally, setting $i=j$ and summing over all index values of i gives the following equation:

$$(3+r\partial_r)\partial_t[c_2\tilde{f}^{ab}(r)]=(3+r\partial_r)\{4[\tilde{a}_{13}^{vba}(r)-\tilde{a}_{12}^{vba}(r)]\}+2\eta(3+r\partial_r)\left[\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right]\times[c_2\tilde{f}^{ab}(r)]. \tag{9}$$

The first integral of this equation is

$$\frac{\partial}{\partial t}[c_2\tilde{f}^{ab}(r)]=4[\tilde{a}_{13}^{vba}(r)-\tilde{a}_{12}^{vba}(r)]+2\eta\left[\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right]\times[c_2\tilde{f}^{ab}(r)]. \tag{10}$$

This is the von Kármán–Howarth equation for magnetic helicity, or VKH-HM. It can be written in a slightly more familiar way, expressing the involved coefficients of the tensors in terms of the parallel (labeled L) and perpendicular (labeled 2 and 3) components, viz.:

$$\tilde{a}_{13}^{vba}(r)=\frac{1}{r}\Phi_{L22}^{vba}(\mathbf{r})=\frac{1}{r}\Phi_{L33}^{vba}(\mathbf{r}),$$

$$\tilde{a}_{12}^{vba}(r)=\frac{1}{r}\Phi_{2L2}^{vba}(\mathbf{r})=\frac{1}{r}\Phi_{3L3}^{vba}(\mathbf{r}),$$

$$c_2\tilde{f}^{ab}(r)=\tilde{R}_{LL}^{H_M}(\mathbf{r});$$

the VKH-HM equation thus also writes

$$\frac{\partial}{\partial t}\tilde{R}_{LL}^{H_M}(\mathbf{r})=\frac{4}{r}[\Phi_{L22}^{vba}(\mathbf{r})-\Phi_{2L2}^{vba}(\mathbf{r})]+2\eta\left[\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right]\tilde{R}_{LL}^{H_M}(\mathbf{r}). \tag{11}$$

Equation (11) is the main exact result of this paper.

B. A simplified form of VKH-HM

The VKH-HM equation (11) can be somewhat simplified in several steps. We first note that the second-order structure function for the magnetic helicity,

$$B_{LL}^{ab}(\mathbf{r})=\langle(a_L(\mathbf{x}')-a_L(\mathbf{x}))(b_L(\mathbf{x}')-b_L(\mathbf{x}))\rangle,$$

can be written in terms of the magnetic helicity correlation function as

$$B_{LL}^{ab}(\mathbf{r})=2\langle a_L(\mathbf{x})b_L(\mathbf{x})\rangle-2\tilde{R}_{LL}^{H_M}(\mathbf{r}). \tag{12}$$

We also define the rate of transfer of magnetic helicity, $\tilde{\epsilon}^{H_M}$, as (see, for example Ref. [10] for the kinetic helicity case)

$$\partial_t\langle a_L(\mathbf{x})b_L(\mathbf{x})\rangle=\frac{1}{3}\partial_t\langle a_i(\mathbf{x})b_i(\mathbf{x})\rangle=-\frac{2}{3}\tilde{\epsilon}^{H_M}. \tag{13}$$

Substituting Eqs. (12) and (13) in Eq. (11) leads to the following alternative form of VKH-HM:

$$-\frac{2}{3}\tilde{\epsilon}^{H_M}-\frac{1}{2}\frac{\partial}{\partial t}B_{LL}^{ab}(\mathbf{r})=\frac{4}{r}[\Phi_{L22}^{vba}(\mathbf{r})-\Phi_{2L2}^{vba}(\mathbf{r})]+2\eta\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\frac{\partial}{\partial r}\tilde{R}_{LL}^{H_M}(\mathbf{r})\right]. \tag{14}$$

C. The long-time, high Reynolds number limit of the VKH-HM equation

The above equivalent Eqs. (10), (11), and (14) are exact. They can be further simplified using now the hypothesis of long-time limit and furthermore neglecting the resistive term in the inertial range, an hypothesis well justified for geophysical and astrophysical flows at high magnetic Reynolds numbers. In that case, one obtains

$$\Phi_{L22}^{vba}(\mathbf{r})-\Phi_{2L2}^{vba}(\mathbf{r})=-\frac{1}{6}\tilde{\epsilon}^{H_M}r, \tag{15}$$

or written explicitly in terms of the basic fields:

$$\langle[v_L(\mathbf{x})b_2(\mathbf{x})-v_2(\mathbf{x})b_L(\mathbf{x})]a_2(\mathbf{x}')\rangle=-\frac{1}{6}\tilde{\epsilon}^{H_M}r. \tag{16}$$

In this form, the (approximate) law (16) is equivalent to that of Kolmogorov [35] for the kinetic energy (see also [36]), as well as those obtained for energy and total cross correlation in MHD [8,9], and for kinetic helicity for the Navier-Stokes equations [10]. Note that Eq. (16) is written in terms of at least two different components of the involved vectors; indeed, because of its helical nature, this relationship cannot be written exclusively in terms of, say, the longitudinal components of the physical fields, contrary to the case of the so-called "four-fifth" law of Kolmogorov [35] which is written in terms of only the longitudinal velocity differences.

Note further that, defining the electromotive force (EMF) due to the turbulent motions $\mathcal{E}^t=\mathbf{v}\times\mathbf{b}$, it is easy to see that the above relationship combines the two distinct normal components of the EMF and of the magnetic potential, viz.,

$$\langle\mathcal{E}_3^t(\mathbf{x})a_2(\mathbf{x}')\rangle=-\frac{1}{6}\tilde{\epsilon}^{H_M}r.$$

A relationship similar to Eq. (16) is obtained when the components with subscript 2 are replaced by the other normal components of index 3; thus, one may equivalently write Eq. (16) as

$$\langle v_L(\mathbf{x})\Sigma_i b_i(\mathbf{x})a_i(\mathbf{x}')\rangle - \langle b_L(\mathbf{x})\Sigma_i v_i(\mathbf{x})a_i(\mathbf{x}')\rangle = -\frac{1}{3}\tilde{\epsilon}^{H_M} r, \tag{17}$$

which now brings it to a form more akin to the law written in Ref. [6] for the passive scalar and in Ref. [36] for kinetic energy (see also Refs. [8–10]). In terms of the electromotive force, this becomes equivalently

$$\langle [\mathcal{E}^t(\mathbf{x})\times \mathbf{a}(\mathbf{x}')]_L\rangle = +\frac{1}{3}\tilde{\epsilon}^{H_M} r. \tag{18}$$

As in all other known cases for these exact laws, the third-order correlators of the physical fields combine together in order to yield a (normal) linear scaling with distance, a result stemming from the invariance law, here of magnetic helicity. As expected, these invariance laws impose correlations between the basic fields, and thus constraints to the dynamics of turbulent flows. For example, anomalous scaling laws followed by structure functions of the velocity and/or the magnetic field have been observed, e.g., with *in situ* measurements in the solar wind [37] and numerically for turbulent MHD flows in both two dimensions [38] and three dimensions [39]; such laws are constrained by the flux relation (17). When measurable (as for direct numerical simulations), the relationship (17) might prove useful, e.g., to define the extent in scale of the inertial range of the inverse cascade of magnetic helicity in an unambiguous way.

IV. CONCLUSION

Three main results are obtained in this paper. On the one hand, we have derived an exact relationship for helical MHD flows, i.e., three-dimensional magnetized flows with no mirror symmetry; it stems from the conservation of magnetic helicity and we call it VKH-HM, or von Kármán–Howarth equation for magnetic helicity. This law is the third exact linear scaling law derived for MHD turbulence and is satisfied simultaneously with the two previous relationships [8,9] obtained for total energy $\langle(\mathbf{v}^2+\mathbf{b}^2)\rangle/2$ and cross correlation $\langle \mathbf{v}\cdot\mathbf{b}\rangle/2$. It obtains from an evaluation of the nonlinear terms in the induction equation, and thus does not arise when the velocity and magnetic field are parallel, in particular in the case of an ensemble of Alfvén waves. Similarly, the aforementioned laws are not applicable for Beltrami and/or force-free MHD flows. This von Kármán–Howarth equation for magnetic helicity, evaluated in the long-time and nondiffusive limit, implies a linear dependence with scale of the correlation between the two different transverse components of the magnetic potential and of the turbulent electromotive force.

The link between the exact law derived in this paper and the dynamo action involving the electromotive force is not clear, in particular in the saturation phase which relies essentially on the conservation of magnetic helicity [21,25]. In light of the law derived here, the problem of the saturation of the dynamo should be examined further using the direct numerical simulations; this point is left for the future work.

Similarly, the role of helicity in the problem of the rate at which the small scale dynamics return to isotropy is a topic of active research presently; we plan to investigate in the future the way in which such a phenomenon in MHD is influenced by the constraint imposed by the VKH-HM equation.

Note further that the VKH-HM equation is independent of the level of equipartition (or lack thereof) between kinetic and magnetic energy. It provides a constraint on the dynamics of turbulent flows that can help their analysis, both numerically and perhaps experimentally, e.g., in the Taylor-Green configuration of counter-rotating disks in gallium or sodium [18,19]. For example, the domain of validity of the law (17) for finite magnetic Reynolds numbers can be chosen as a nonambiguous definition, at least for low to moderate orders of correlators, of the inertial range of magnetic helicity. The difficulty of measuring the magnetic potential experimentally can be alleviated by using Eq. (11) and rewriting it in terms of the magnetic field only (and the velocity), following the work in Ref. [10] for kinetic helicity (see also Ref. [40]). Indeed, using the fact that $r\epsilon_{23L}\Phi^{vbb}_{23L}(\mathbf{r}) = \Phi^{vba}_{L22} - \Phi^{vba}_{2L2}$, one can readily show that the inertial range relationship (16) becomes, in terms of the correlations between velocity and magnetic field,

$$\langle v_2(\mathbf{x})b_3(\mathbf{x})b_L(\mathbf{x}')\rangle = -\langle v_3(\mathbf{x})b_2(\mathbf{x})b_L(\mathbf{x}')\rangle = -\frac{1}{6}\tilde{\epsilon}^{H_M}.$$

This implies the constancy of the correlator involving the electromotive force $\mathcal{E}^t$, viz.:

$$\langle \mathcal{E}^t_L(\mathbf{x})b_L(\mathbf{x}')\rangle = -\frac{1}{3}\tilde{\epsilon}^{H_M}.$$

Finally, at a more technical level, the general functional form of the third-order isotropic but nonmirror symmetric correlation tensors between the three physical fields (either vectors or pseudovectors) for homogeneous incompressible flows is given in the Appendix. It is shown that four defining functions are necessary in general, reducing to only two functions when two of the three vectors are identical, as is the case for kinetic helicity in Navier-Stokes flows.

This implies that, at the level of third-order correlations, four independent functions are necessary, in general, to characterize fully the turbulent dynamics of the flow, leading *a priori* to four different scaling laws that will likely have to be unraveled using the direct numerical simulations. This requires highly resolved computations in three dimensions, which will be the outcome of a future work. The case of kinetic helicity, involving only two functions, may be simpler yet and should be taken as a first step toward checking whether multiscaling occurs in helical flows. This point, although difficult, could indeed be examined on numerical or observational data, e.g., in the planetary boundary layer at very high Taylor-Reynolds number.

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APPENDIX: CORRELATION TENSORS IN HELICAL FLOWS

The kinematics of isotropic tensors has been elaborated in several studies [29,30,33,41] for special cases, but the most general form of the third-order tensors in a homogeneous isotropic but nonmirror symmetric flow has not been written to our knowledge. We also mention the case of tensors involving axial vectors such as in MHD or when dealing with the vorticity, since one must then *a priori* distinguish between pseudoscalar and true scalar functions entering in the expression of such tensors.

Thus this Appendix is intended to be slightly more general than what is needed for treating the magnetic helicity case. We give a complete exposition of the form such tensors can take, following the well-known ideas already presented in Robertson [33] and in Batchelor [30]. For the anisotropic case, see, e.g., [41,42], and more recently Ref. [43].

1. Second-order tensors

For the sake of completeness, we first give the general expressions of second-order tensors, as can be found in Oughton *et al.* [29].

The two-points correlation tensor between two arbitrary *vectors* **u** and **v** is a proper (true) tensor and can be written as

$$R_{ij}^{uv}(\mathbf{r}) = \langle u_i(\mathbf{x}) v_j(\mathbf{x}')\rangle = A^{uv}(r) r_i r_j + B^{uv}(r)\delta_{ij} + \tilde{C}^{uv}(r)\epsilon_{ij\ell} r_\ell, \quad \text{(A1)}$$

where $\mathbf{x}' = \mathbf{x} + \mathbf{r}$, and the two functions $A^{uv}(r)$ and $B^{uv}(r)$ are scalars and $\tilde{C}^{uv}(r)$ is pseudoscalar. All three are even functions of the distance r because of isotropy, i.e., spherical symmetry. Moreover, $R_{ij}^{uv}(\mathbf{r}) = R_{ji}^{vu}(-\mathbf{r})$ because of homogeneity. When both **u** and **v** are equal to the velocity field, A^{uu} and B^{uu} are related to the energy of the flow and $\tilde{C}^{uu}$ to the kinetic helicity $\langle \mathbf{u}\cdot\boldsymbol{\omega}\rangle/2$. Note that when $R_{ij}^{uv}(\mathbf{r})$ is a pseudotensor [by having either **u** or (exclusive) **v** as a pseudovector], the same expression as in Eq. (A1) applies, but now both $A^{uv}(r)$ and $B^{uv}(r)$ are pseudoscalars and $\tilde{C}^{uv}(r)$ is a scalar.

Incompressibility implies that

$$\frac{\partial R_{ij}^{uv}(\mathbf{r})}{\partial r_j} = 0,$$

which in turn leads to a relationship between A^{uv} and B^{uv} in dimension d, namely:

$$(d+1)A^{uv}(r) + r\frac{\partial A^{uv}(r)}{\partial r} + \frac{1}{r}\frac{\partial B^{uv}(r)}{\partial r} = 0.$$

We define further the longitudinal and helical correlation functions $f^{uv}(r)$ and $\tilde{s}^{uv}(r)$ as

$$R_{LL}^{uv}(\mathbf{r}) = c_2 f^{uv}(r), \quad R_{23}^{uv}(\mathbf{r}) = c_2 \tilde{s}^{uv}(r),$$

where L refers to the direction along **r**, and 2 and 3 to the two normal directions. We thus have

$$R_{LL}^{uv}(\mathbf{r}) = A^{uv}(r) r^2 + B^{uv}(r) = c_2 f^{uv}(r), \quad \text{(A2)}$$

$$R_{23}^{uv}(\mathbf{r}) = \tilde{C}^{uv}(r) r \epsilon_{L23} = c_2 \tilde{s}^{uv}(r)\epsilon_{L23}, \quad \text{(A3)}$$

$$R_{22}^{uv}(\mathbf{r}) = R_{33}^{uv}(\mathbf{r}) = B^{uv}(r) = c_2 g^{uv}(r), \quad \text{(A4)}$$

where $c_2 = \overline{uv}$ is a normalization coefficient using the rms fields $\overline{u}$ and $\overline{v}$. Note that $f^{uv}(r)$ and $\tilde{s}^{uv}(r)$ are the two generating functions of the second-order tensor, and they are, respectively even and odd functions of the distance r. The third function $g^{uv}(r)$ can be eliminated using the incompressibility condition, viz.,

$$g^{uv}(r) = f^{uv}(r) + \frac{r}{d-1}\partial_r f^{uv}(r),$$

with $\partial_r f^{uv}(r) = \partial f^{uv}/\partial r$.

We can now write the following standard expression for a second-order tensor in the isotropic helical case:

$$R_{ij}^{uv}(\mathbf{r}) = c_2\left[f^{uv}(r)\delta_{ij} + \frac{r}{d-1}\frac{\partial f^{uv}(r)}{\partial r}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right) + \epsilon_{ij\ell}\frac{r_\ell}{r}\tilde{s}^{uv}(r)\right].$$

For $i=j$ and $d=3$, one recovers the usual relationship:

$$R_{ii}^{uv}(\mathbf{r}) = (3 + r\partial_r)[c_2 f^{uv}(r)].$$

As a special example taken from Oughton *et al.* [29], we write the two pseudotensors as

$$R_{ij}^{\pm}(\mathbf{r}) = \langle v_i(\mathbf{x}) b_j(\mathbf{x}') \pm b_i(\mathbf{x}) v_j(\mathbf{x}')\rangle$$

related to the cross correlation of the velocity and the magnetic field; they each have a symmetric and an antisymmetric part, namely:

$$R_{ij}^{+}(\mathbf{r}) = \frac{1}{2}[\langle v_i(\mathbf{x}) b_j(\mathbf{x}') + b_i(\mathbf{x}) v_j(\mathbf{x}')\rangle + \langle v_j(\mathbf{x}) b_i(\mathbf{x}') + b_j(\mathbf{x}) v_i(\mathbf{x}')\rangle]$$

and a similar expression is written for R_{ij}^{-}.

Another example is written below for the magnetic helicity tensor, central to the present paper; identifying **u** with the magnetic potential **a** and **v** with the magnetic induction $\mathbf{b} = \boldsymbol{\nabla}\times\mathbf{a}$, we have

$$R_{ij}^{ab}(\mathbf{r}) = \epsilon_{j\ell m}\partial R_{im}^{aa}(\mathbf{r})/\partial r_\ell = \tilde{R}_{ij}^{H_M}(\mathbf{r}).$$

The relationships between the (aa) and (ab) tensor coefficients—following the definitions in Eq. (A1)—are

$$\tilde{A}^{ab}(r)=\frac{1}{r}\partial_r\tilde{C}^{aa}(r), \tag{A5}$$

$$\tilde{B}^{ab}(r)=-[r\partial_r\tilde{C}^{aa}(r)+2\tilde{C}^{aa}(r)]=-\frac{1}{r}\partial_r[r^2\tilde{C}^{aa}(r)], \tag{A6}$$

$$C^{ab}(r)=-A^{aa}(r)+\frac{1}{r}\partial_r B^{aa}(r). \tag{A7}$$

For $i=j$ and at $r=0$, one recovers

$$H^M=\frac{1}{2}\langle\mathbf{a}\cdot\mathbf{b}\rangle=-3\tilde{C}^{aa}(0). \tag{A8}$$

2. Third-order correlation tensors

We now begin the derivation of the functional form of third-order tensors in the isotropic case in helical geometry.

a. The general case

We define a general third-order correlator tensor between three *vectorial* fields $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as

$$\Phi^{uvw}_{ij\ell}(\mathbf{r})=\langle u_i(\mathbf{x})v_j(\mathbf{x})w_\ell(\mathbf{x}')\rangle,$$

with $\mathbf{x}'=\mathbf{x}+\mathbf{r}$ and with homogeneity assumed. Since isotropy is also assumed, i.e., spherical symmetry or invariance under arbitrary rigid rotations, but not reflexional (or mirror), symmetry, the third-order two-point correlation tensor can be *a priori* written as

$$\Phi^{uvw}_{ij\ell}(\mathbf{r})=a_{11}(r)r_\ell\delta_{ij}+a_{12}(r)r_j\delta_{i\ell}+a_{13}(r)r_i\delta_{j\ell}+a_3(r)r_ir_jr_\ell+\tilde{b}_{23}(r)r_i\epsilon_{j\ell m}r_m+\tilde{b}_{22}(r)r_j\epsilon_{\ell im}r_m+\tilde{b}_{21}(r)r_\ell\epsilon_{ijm}r_m, \tag{A9}$$

where the seven tensor coefficients $a_{\ldots}$ and $\tilde{b}_{\ldots}$ are even functions of the distance r. Note that the superscript uvw in the seven coefficients is omitted for the sake of simplicity. In expressing this formulation of the tensor, the relationship

$$r_i\epsilon_{j\ell m}r_m+r_j\epsilon_{\ell im}r_m+r_\ell\epsilon_{ijm}r_m=r^2\epsilon_{ij\ell}$$

is found useful in eliminating one term in Eq. (A9), the one proportional to $\epsilon_{ij\ell}$.

As in the case of the second-order tensor, we can define correlation functions in terms of the longitudinal and the normal components of the involved fields, namely:

$$\Phi^{uvw}_{LLL}(\mathbf{r})=c_3k(r)=r[a_{11}(r)+a_{12}(r)+a_{13}(r)+r^2a_3(r)], \tag{A10}$$

$$\Phi^{uvw}_{L22}(\mathbf{r})=c_3q_3(r)=ra_{13}(r)=\Phi^{uvw}_{L33}(\mathbf{r}), \tag{A11}$$

$$\Phi^{uvw}_{2L2}(\mathbf{r})=c_3q_2(r)=ra_{12}(r)=\Phi^{uvw}_{3L3}(\mathbf{r}), \tag{A12}$$

$$\Phi^{uvw}_{22L}(\mathbf{r})=c_3h(r)=ra_{11}(r)=\Phi^{uvw}_{33L}(\mathbf{r}), \tag{A13}$$

$$\Phi^{uvw}_{L23}(\mathbf{r})=c_3\tilde{\alpha}_3(r)\epsilon_{23L}=r^2\tilde{b}_{23}(r)\epsilon_{23L}=-\Phi^{uvw}_{L32}(\mathbf{r}), \tag{A14}$$

$$\Phi^{uvw}_{3L2}(\mathbf{r})=c_3\tilde{\alpha}_2(r)\epsilon_{23L}=r^2\tilde{b}_{22}(r)\epsilon_{23L}=-\Phi^{uvw}_{2L3}(\mathbf{r}), \tag{A15}$$

$$\Phi^{uvw}_{23L}(\mathbf{r})=c_3\tilde{\beta}(r)\epsilon_{23L}=r^2\tilde{b}_{21}(r)\epsilon_{23L}=-\Phi^{uvw}_{32L}(\mathbf{r}), \tag{A16}$$

where $c_3=\overline{uvw}$ is a (scalar) normalization coefficient based on the (scalar) r.m.s. values of the three fields, so that the defining functions of the tensors be dimensionless. One can easily see that $\tilde{\alpha}_2(r)$, $\tilde{\alpha}_3(r)$, and $\tilde{\beta}(r)$ are even functions of r, while $k(r)$, $q_2(r)$, $q_3(r)$, and $h(r)$ are odd functions of r.

Obviously, if one (and only one) field from the triplet $(\mathbf{u},\mathbf{v},\mathbf{w})$ *is not* a vector, this in turn exchanges the scalar or pseudoscalar nature of the $a_{ij}(r)$ and $\tilde{b}_{ij}(r)$ functions. Moreover, note that

$$\langle u_i(\mathbf{x})v_j(\mathbf{x})w_\ell(\mathbf{x}-\mathbf{r})\rangle=\langle u_i(\mathbf{x}')v_j(\mathbf{x}')w_\ell(\mathbf{x})\rangle$$

under the homogeneity assumption.

Two examples of third-order tensors relevant to the present study of magnetic helicity are $\Phi^{vba}_{ij\ell}(\mathbf{r})=\langle v_i(\mathbf{x})b_j(\mathbf{x})a_\ell(\mathbf{x}')\rangle$, which is a pseudotensor, and $\Phi^{vbb}_{ij\ell}(\mathbf{r})=\langle v_i(\mathbf{x})b_j(\mathbf{x})b_\ell(\mathbf{x}')\rangle$, which is a true tensor; where $\mathbf{v}$ is the velocity field, $\mathbf{b}$ the magnetic induction, and $\mathbf{a}$ its vector potential.

It can be found useful to decompose the tensor $\Phi^{uvw}_{ij\ell}(\mathbf{r})$ into its even, P, and an odd, Q, parts, viz.:

$$P^{uvw}_{ij\ell}(\mathbf{r})=\frac{1}{2}[\Phi^{uvw}_{ij\ell}(\mathbf{r})+\Phi^{uvw}_{ij\ell}(-\mathbf{r})], \tag{A17}$$

$$Q^{uvw}_{ij\ell}(\mathbf{r})=\frac{1}{2}[\Phi^{uvw}_{ij\ell}(\mathbf{r})-\Phi^{uvw}_{ij\ell}(-\mathbf{r})]. \tag{A18}$$

This is done independently of the fact that the tensor may be symmetric $(+)$ or anti-symmetric $(-)$ in its first two indices, viz., $\Phi^{uvw}_{ij\ell}(\mathbf{r})=\pm\Phi^{uvw}_{ji\ell}(\mathbf{r})$. Indeed, one defines the symmetric $S^{uvw}_{ij\ell}(\mathbf{r})$ and antisymmetric $A^{uvw}_{ij\ell}(\mathbf{r})$ part of a tensor as usual:

$$\Phi^{uvw}_{ij\ell}(\mathbf{r})=S^{uvw}_{ij\ell}(\mathbf{r})+A^{uvw}_{ij\ell}(\mathbf{r}), \tag{A19}$$

with

$$S^{uvw}_{ij\ell}(\mathbf{r})=\frac{1}{2}[\Phi^{uvw}_{ij\ell}(\mathbf{r})+\Phi^{uvw}_{ji\ell}(\mathbf{r})]$$

and

$$A^{uvw}_{ij\ell}(\mathbf{r})=\frac{1}{2}[\Phi^{uvw}_{ij\ell}(\mathbf{r})-\Phi^{uvw}_{ji\ell}(\mathbf{r})].$$

When symmetries with respect to indices occur, i.e., when $\mathbf{u}\equiv\mathbf{v}$, relationships between the seven functions defined above can be easily derived, even without using yet at this stage the incompressibility condition.

Finally, note that only in the full isotropic case (which we define as the case without helicity, i.e., with mirror symmetry), can we write the homogeneity relationship as

$$\langle u_\alpha(\mathbf{x})u_\beta(\mathbf{x})u_\gamma(\mathbf{x}-\mathbf{r})\rangle = -\langle u_\alpha(\mathbf{x})u_\beta(\mathbf{x})u_\gamma(\mathbf{x}+\mathbf{r})\rangle.$$

b. The incompressible case

For incompressible flows, the following continuity condition must be satisfied:

$$\frac{\partial \Phi^{uvw}_{ij\ell}(\mathbf{r})}{\partial r_\ell}=0,$$

for all values of $\mathbf{r}$; it yields

$$r\frac{\partial a_{11}}{\partial r}+da_{11}+(a_{12}+a_{13})=0, \tag{A20}$$

$$\frac{1}{r}\left(\frac{\partial a_{12}}{\partial r}+\frac{\partial a_{13}}{\partial r}\right)+r\frac{\partial a_3}{\partial r}+(d+2)a_3=0, \tag{A21}$$

$$-(\tilde{b}_{23}+\tilde{b}_{22})+(d+1)\tilde{b}_{21}+r\frac{\partial \tilde{b}_{21}}{\partial r}=0. \tag{A22}$$

Furthermore, by contraction on its first two indices, $\Phi^{uvw}_{ij\ell}(\mathbf{r})$ reduces to a solenoidal first-order tensor $\Phi^{uvw}_{ii\ell}(\mathbf{r})$ which is therefore identically zero, assuming as usual the absence of nonregular solutions at $\mathbf{r}=0$. Thus, $\partial\Phi^{uvw}_{ii\ell}(\mathbf{r})/\partial r_\ell=0$, which leads to

$$da_{11}+(a_{12}+a_{13})+a_3r^2=0.$$

With these relationships, the seven functions appearing in the definition of the general third-order tensor can now be written in terms of only four generating functions, which we choose to be $k(r)$, $q_2(r)$, $\tilde{\beta}(r)$, and $\tilde{\alpha}_2(r)$; the last two being identically zero in the full isotropic case. As can be seen below, they are simply related to $a_{11}(r)$, $a_{12}(r)$, $\tilde{b}_{21}(r)$, and $\tilde{b}_{22}(r)$ through

$$a_{11}(r)=-c_3\frac{k(r)}{r(d-1)}, \tag{A23}$$

$$a_{12}(r)=c_3\frac{q_2(r)}{r}, \tag{A24}$$

$$\tilde{b}_{21}(r)=c_3\frac{\tilde{\beta}(r)}{r^2}, \tag{A25}$$

$$\tilde{b}_{22}(r)=c_3\frac{\tilde{\alpha}_2(r)}{r^2}, \tag{A26}$$

with the remaining three functions being given by

$$a_3(r)=c_3\frac{k(r)-r\partial_r k(r)}{r^3(d-1)}, \tag{A27}$$

$$a_{13}(r)=c_3\left[\frac{-q_2(r)}{r}+\frac{(d-1)k(r)+r\partial_r k(r)}{r(d-1)}\right], \tag{A28}$$

$$\tilde{b}_{23}(r)=c_3\left[\frac{(d-1)\tilde{\beta}(r)+r\partial_r\tilde{\beta}(r)-\tilde{\alpha}_2(r)}{r^2}\right]; \tag{A29}$$

d is the space dimension which, in the helical case, we take equal to 3.

For third-order tensors, which are symmetric in their first two indices, the four generating functions above can be reduced to only two, one scalar and one pseudoscalar (as was already known for the case of kinetic helicity); we can choose, e.g., $k(r)$ and $\tilde{\alpha}_2(r)$. Moreover, when full isotropy is assumed, only one generating function remains, namely $k(r)$ which identifies with the classical longitudinal correlation function for neutral fluids.

3. Structure functions

We turn now to second- and third-order tensors built on the increments of components of vector (or pseudovector) fields, $u_i(\mathbf{x}+\mathbf{r})-u_i(\mathbf{x})$. At second order, it reads

$$B^{uv}_{ij}(\mathbf{r})\equiv\langle(u_i(\mathbf{x}')-u_i(\mathbf{x}))(v_j(\mathbf{x}')-v_j(\mathbf{x}))\rangle,$$

with straightforwardly

$$B^{uv}_{ij}(\mathbf{r})=2\langle(u_i(\mathbf{x})v_j(\mathbf{x}))\rangle-R^{uv}_{ij}(\mathbf{r})-R^{uv}_{ij}(-\mathbf{r})$$
$$=2\langle(u_i(\mathbf{x})v_j(\mathbf{x}))\rangle-2P^{uv}_{ij}(\mathbf{r}).$$

Here, one has written $R^{uv}_{ij}(\mathbf{r})=P^{uv}_{ij}(\mathbf{r})+Q^{uv}_{ij}(\mathbf{r})$, where $P^{uv}_{ij}(\mathbf{r})=A^{uv}(r)r_ir_j+B^{uv}(r)\delta_{ij}$ is the even part of the tensor and $Q^{uv}_{ij}(\mathbf{r})=\tilde{C}^{uv}(r)\epsilon_{ij\ell}r_\ell$ its odd part. Note that, in this case, the even part of $R^{uv}_{ij}(\mathbf{r})$ is also its symmetric part and its odd part corresponds to its antisymmetric part.

Similarly, the third-order cross-structure function between $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is defined as

$$B^{uvw}_{ij\ell}(\mathbf{r})\equiv\langle(u_i(\mathbf{x}')-u_i(\mathbf{x}))(v_j(\mathbf{x}')-v_j(\mathbf{x}))\times(w_\ell(\mathbf{x}')-w_\ell(\mathbf{x}))\rangle; \tag{A30}$$

by means of the homogeneity assumption, it can be written as

$$B_{ij\ell}^{uvw}(\mathbf{r})=[\Phi_{j\ell i}^{vwu}(\mathbf{r})-\Phi_{j\ell i}^{vwu}(-\mathbf{r})]+[\Phi_{ij\ell}^{uvw}(\mathbf{r})-\Phi_{ij\ell}^{uvw}(-\mathbf{r})]+[\Phi_{i\ell j}^{uwv}(\mathbf{r})-\Phi_{i\ell j}^{uwv}(-\mathbf{r})].$$

With the definitions given in Eq. (A18), it simplifies to

$$B_{ij\ell}^{uvw}(\mathbf{r})=2[Q_{ij\ell}^{uvw}(\mathbf{r})+Q_{i\ell j}^{uwv}(\mathbf{r})+Q_{j\ell i}^{vwu}(\mathbf{r})], \quad \text{(A31)}$$

with $Q_{ij\ell}^{uvw}(\mathbf{r})$ the odd part of the $\Phi_{ij\ell}^{uvw}(\mathbf{r})$ correlation tensor. This relationship is a generalization of the classical Navier-Stokes (fluid) law,

$$B_{LLL}^{uuu}(\mathbf{r})=6Q_{LLL}^{uuu}(\mathbf{r}),$$

between longitudinal correlation and structure functions of the velocity field (see, e.g., Ref. [44]).

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